Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Engineering Analysis

Chapter 5
Partial Differential Equations

Assistant Prof. Dr. Eng. Ibrahim Thamer Nazzal

Differential Equation

Differential Equation: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

Classification of **differential equations**

Ordinary Differential Equation (ODE): If an equation contains only ordinary derivatives of one or more dependent variables w.r.t. a single independent variable, it is said to be an ODE.

Example: $$\frac{dy}{dx} + 5y = e^{x}$$

Partial Differential Equation (PDE): An equation involving the partial derivatives of one or more dependent variables w.r.t. two or more independent variable is called a **PDE**

Example: $$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} \qquad \left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial y}\right)$$

The order of a partial differential equation is the order of the highest derivative involved. A solution (or a particular solution) to a partial differential equation is a function that solves the equation or, in other words, turns it into an identity when substituted into the equation.

Heat Conduction Equation (1 – D): $$\frac{\partial T}{\partial t} = C\frac{\partial^2 T}{\partial x^2}$$

Classification of Partial Differential Equations (PDEs)

There are 6 basic classifications:

(1) Order of PDE

(2) Number of independent variables

(3) Linearity

(4) Homogeneity

(5) Types of coefficients

(6) Canonical forms for 2nd order PDEs

(1) Order of PDEs

The order of a PDE is the order of the highest partial derivative in the equation.

Examples:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

(1st order)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(2nd order)

$$\frac{\partial u}{\partial t} = u\frac{\partial^3 u}{\partial x^3} + \sin x$$

(3rd order)

(2) Number of Independent Variables

Examples:

(2 variables: x and t)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(3 variables: r, θ, and t)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

(3) Linearity

PDEs can be linear or non-linear. A PDE is linear if the dependent variable and all its derivatives appear in a linear fashion (i.e. they are not multiplied together or squared for example.

Examples:

(Linear) $$\frac{\partial^2 u}{\partial t^2} = e^{-t}\frac{\partial^2 u}{\partial x^2} + \sin t$$

(Non-linear) $$u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$$

(Linear) $$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$$

(Non-linear) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + u^2 = 0$$

(Linear) $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \sin x$$

(Non-linear) $$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \sin u = e^y$$

(Non-linear) $$\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial u}{\partial y} = 1$$

(4) Homogeneity

A PDE is called homogenous if after writing the terms in order, the right hand side is zero.

Examples:

(Non-homogeneous) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

(Non-homogeneous) $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=u-5$$

(Homogeneous) $$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=0$$

(Homogenous) $$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=u$$

(Homogeneous) $$\frac{\partial (u-5)}{\partial x}+\frac{\partial (u-5)}{\partial t}=u-5$$

(5) Types of Coefficients

If the coefficients in front of each term involving the dependent variable and its derivatives are independent of the variables (dependent or independent), then that PDE is one with constant coefficients.

Examples

(Variable coefficients) $$\frac{\partial^2 u}{\partial x^2}+x^2\frac{\partial^2 u}{\partial y^2}=0$$

(C constant; constant coefficients) $$\frac{\partial^2 u}{\partial x^2}-C\frac{\partial^2 u}{\partial t^2}=0$$

(6) Canonical forms for 2nd order PDEs (Linear)

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G \qquad \text{(Standard Form)}$$

where A, B, C, D, E, F, and G are either real constants or real-valued functions of x and/or y.

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

This terminology of elliptic, parabolic, and hyperbolic, reflect the analogy between the standard form for the linear, 2nd order PDE and conic sections encountered in analytical geometry:

$B^2-4AC<0 \Rightarrow$ **PDE is <u>Elliptic</u>**

$B^2-4AC=0 \Rightarrow$ **PDE is <u>Parabolic</u>**

$B^2-4AC>0 \Rightarrow$ **PDE is <u>Hyperbolic</u>**

Parabolic PDE $\Rightarrow$ solution "propagates" or diffuses

Hyperbolic PDE $\Rightarrow$ solution propagates as a wave

Elliptic PDE $\Rightarrow$ equilibrium

Examples

(a) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = 0$ Here, A=1, B=0, C=2, D=E=F=G=0 $\Rightarrow$

B^2-4AC = 0 - 4(1)(2) = -8 < 0 $\Rightarrow$ this equation is elliptic.

(b) $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = 0$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Here, A=1, B=0, C=-2, D=E=F=G=0 $\Rightarrow$ B^2-4AC = 0 - 4(1)(-2) = 8 > 0 $\Rightarrow$ this equation is hyperbolic.

(c) $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial y} = 0$

Here, A=1, B=0, E=-2, C=D=F=G=0 $\Rightarrow$ B^2-4AC = 0 - 4(1)(0) = 0 $\Rightarrow$ this equation is parabolic.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

Solution of Partial Differential Equations

Separation of Variable Solutions

Separation of variables is a technique for solving some partial deferential equations.

Assume the function you're looking for, u(x; t), can be written as a product of a function of x only and a function of t only:

$$u(x;\ t) = X(x)\,T(t)$$

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} - \frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{0}$$

Then it is easy to take derivatives:

$$u_x = \acute{X}(x)T(t) \qquad u_{xx} = X''(x)T(t)$$

$$u_t = X(x)T'(t) \qquad u_{tt} = X(x)T''(t)$$

Plug them in to the partial deferential equation.

Try to separate the variables: (function of *x* only) = (function of *t* only)

If you can, then both sides must be constant: (function of *x* only) = λ = (function of *t* only)

Reorganize these into two ordinary deferential equations

function of x only) = λ ((function of t only) = λ

which you can solve separately for *X* and *T*.

Example 1

Use separation of variables to convert the following partial deferential equation into two ordinary deferential equations:

$$u_{xx} + x\,u_t = 0$$

$$u(x;\ t) = X(x)T(t)$$

$$u_x = X'(x)\,T(t)$$

$$u_{xx} = X''(x)\,T(t)$$

$$u_t = X(x)\,T'(t)$$

Plug in to the PDE: $X''(x)\,T(t) + x\,X(x)T'(t) = 0$

$$-\frac{X''(x)}{x\,X(x)} = \frac{T'(t)}{T(t)} = \lambda$$

$$X''(x) + \lambda\,x\,X(x) = 0$$

$$T'(t) - \lambda T(t) = 0$$

Example 2

Use separation of variables to convert the following partial differential equation into two ordinary deferential equations:

$$u_{tt} + u_{xt} + u_x = 0$$

$$u(x;\ t) = X(x)T(t)$$

$$u_x = X'(x)\,T(t)$$

$$u_{tt} = X(x)T''(x)$$

$$u_{xt} = X'(x)T'(t)$$

$$Plug\ in\ to\ the\ PDE:\ X(x)T''(x) + X'(x)T'(t) + X'(x)T(t) = 0$$

$$X(x)T''(x) + X'(x)[T'(t) + T(t)] = 0$$

$$X'(x)[T'(t) + T(t)] = -X(x)T''(x)$$

$$-\frac{X'(x)}{X(x)} = \frac{T''(t)}{T'(t) + T(t)} = \lambda$$

$$X'(x) + \lambda X(x) = 0$$

$$T''(t) - \lambda T'(t) \text{-} \lambda T(t) = 0$$

Example 3 Solve by the separation of variables $3u_x+u_y = 0\,, given\ that\ u(x,o) = 4e^{-x}$

Solution given $3\frac{\partial u}{\partial x} + 2\ \frac{\partial u}{\partial y} = 0$,

$u(x;y) = X\,Y$ (1) $where\ u(x;\ t) = X(x) = X\ \ and\ Y(y) = Y$

$$3\frac{\partial}{\partial x}(XY) + 2\ \frac{\partial}{\partial y}(XY) = 0 \quad \Longrightarrow \frac{3}{X}\frac{dX}{dx} = -\frac{2}{Y}\ \frac{dY}{dy}$$

$$\Longrightarrow \frac{3}{X}\frac{dX}{dx} = \mathrm{k} \quad \Longrightarrow \quad 3\ \frac{dX}{X} = kdx$$

$$\Longrightarrow\ Ln\ X\ = \frac{kx}{3} + c_1$$

$$X = e^{\frac{kx}{3}+c_1}$$

$$-\frac{2}{Y}\ \frac{dY}{dy} = k \Longrightarrow \frac{dY}{Y} = -\ \frac{k}{2}dy \qquad Ln\ Y\ = -\ \frac{ky}{2} + c_2 \qquad Y = e^{\frac{-ky}{2}+c_2}$$

Substitute X and Y in (1)

$$U = e^{k\,(\frac{x}{3} - \frac{y}{2})+c_1+\,c_2}$$

$$also\ u(x,o) = 4e^{-x}$$

$$4e^{-x} = e^{k\,\frac{x}{3}\ +c_1+\,c_2} = Ae^{k\,\frac{x}{3}}$$

so $A = 4$ and $k = -3$ $\Longrightarrow$ $U = 4\,e^{-3\,(\frac{x}{3} - \frac{y}{2})}$

Example 4

Use separation of variables to convert the heat equation below into two ordinary differential equations. (For later purposes, use $-\lambda$ instead of λ for the separation constant.)

$$u_t = \alpha^2 \ u_{xx}$$

$$u(x;\ t) \ = \ X(x)T(t)$$

$$u_x = X'(x)\,T(t)$$

$$u_{xx} = X''(x)\,T(t)$$

$$u_t = X(x)T'(t)$$

Plug in to the PDE: $u_t = \alpha^2\, u_{xx}$ $\Rightarrow$ $X(x)T'(t) = \alpha^2\, X''(x)\,T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2\, T(t)} = -\lambda$$

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$X'(x) + \ \lambda\, X(x) \ = \ 0$$

$$\frac{T'(t)}{\alpha^2\, T(t)} = -\lambda \quad \Rightarrow \quad T'(t) + \lambda\alpha^2\, T(t) = 0$$

1D Heat equation

Consider 1D heat equation of the form $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. This function is defined on the spatial domain $0 \le x \le L$ and $t > 0$. BCs: $u(0,t) = u(L,t) = 0$, IC: $u(x,0) = f(x)$. Solve the equation using separation of variable

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}. \qquad u_t = \alpha^2 u_{xx}$$

$$u(x;\, t) = X(x)T(t)$$

$$u_x = X'(x)\,T(t) \qquad u_{xx} = X''(x)\,T(t)$$

$$u_t = X(x)\;\; T'(t)$$

Now, substituting these expression into $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. $\Rightarrow$ $X(x)T'(t) = \alpha^2 X''(x)\,T(t)$

Separating variables, we obtain

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = k$$

$$\frac{X''(x)}{X(x)} = k \qquad X''(x) - k\,X(x) = 0$$

$$\frac{T'(t)}{\alpha^2 T(t)} = k \qquad T'(t) - k\alpha^2\,T(t) = 0$$

. There are three distinct cases affecting general solution.

$k < 0$ (Negative coefficient) $k = 0$ (Null coefficient) $k > 0$ (positive coefficient)

It is convenient to set $k = -\lambda^2$ for $k < 0$ and $k = \lambda^2$ for $k > 0$

$$X''(x) - k\,X(x) = 0$$

Case 1) What if K = 0 ? Then it follows that

$$T'(t) = 0 \qquad X''(x) = 0$$

$$X''(x) - k\,X(x) = 0$$

$$T'(t) - k\alpha^2\,T(t) = 0$$

The general solutions are :

$$T(t) = A1 \qquad and \;\; X(x) = B1\,x + C1$$

thus $u = XT = A1\,(B1\,x + C1)$

Applying BC which $u(0,t) = u(L,t) = 0 = X(0,t) = X(L,t) = 0$ yields

We get that $B1 = C1 = 0$ This lead to $u = 0$

This trivial solution we reject $k = -\lambda^2 = 0$

Case 2) $if\;k > 0 = \lambda^2 \quad then \quad k = \lambda^2$ and thus

$$X''(x) - k\,X(x) = 0$$

$$X''(x) - \lambda^2 X(x) = 0 \qquad T'(t) - \lambda^2\,\alpha^2\,T(t) = 0$$

which yields the following parametrized general solutions

$$T(t) = A2\,e^{\lambda^2\alpha^2 t} \qquad X(x) = B2\,e^{\lambda x} + C2e^{-\lambda x}$$

thus $u(x;\,t) = X(x)\;T(t) = A2\;\;e^{\alpha^2\lambda^2 t}\,(B2\,e^{\lambda x} + C2e^{-\lambda x})$

Applying BC which $u(0,t) = u(L,t) = 0 = X(0,t) = X(L,t) = 0$ yields

To determine $B2$ and $C2$, we apply the BC $X(0) = 0$, $X(L) = 0$ to have

$X(x) = B2\, e^{\lambda x} + C2 e^{-\lambda x}$ $\quad X(0) = B2 + C2 = 0,$ $\quad \Rightarrow \quad C2 = -B2$

$X(L) = B\, e^{\lambda L} + C e^{-\lambda L} = 0$

$B2\,(e^{\lambda L} - e^{-\lambda L}) = 0$ as $\lambda > 0$.

$B2\,(e^{2\lambda L} - 1) = 0$ $\quad \Rightarrow \quad B2 = 0$ $\quad$ *Thus* $C2 = 0$

this implies $u(x, t) = 0$ $\quad$ this trivial solution we reject $k > 0$

Case 3) if $k < 0 = -\lambda^2$ $\quad X''(x) - k\,X(x) = 0$

$X''(x) + \lambda^2 X(x) = 0$ $\quad T'(t) + \lambda^2 \alpha^2 T(t) = 0$ $\quad X''(x) - \lambda^2 X(x) = 0$

The general solution

$T(t) = A3\, e^{-\lambda^2 \alpha^2}$ $\quad X(x) = B3 \cos(\lambda x) + C3 \sin((\lambda x)$

thus $u(x;\, t) = X(x)T(t) = A3\ e^{-\alpha^2 \lambda^2 t}\,(B3 \cos(\lambda x) + C3 \sin(\lambda x))$

To determine $B3$ and $C3$, we apply the BC $X(0) = 0$, $X(L) = 0$ to have

$X(0) = 0 = B3\,(1) + 0$ $\quad$ this lead B3 = 0

$X(L) = B3\cos(\lambda L) + C3\sin(\lambda L) = 0$

As B3 = 0 the equation reduces to $C3\sin((L\lambda) = 0$

for this case $C3 = 0$ or $\sin(\lambda L) = 0$

we reject C3 = 0 because of trivial solution.

so sin $(\lambda L) = 0 \Longrightarrow \lambda L = n\pi$ *or* $\lambda = \frac{n\pi}{L}$

$X(x) = B3\cos(\lambda x) + C3\sin(\lambda x)$

$X_n(x) = Cn\sin\left(\frac{n\pi}{L}x\right)$ $T(t) = an\ e^{-\alpha^2\frac{n^2\pi^2}{L^2}t}$

$$u(x;\ t)\ \sum_{n=1}^{\infty} u(x;\ t) = \sum_{n=1}^{\infty} cn\ e^{-\alpha^2\frac{n^2\pi^2}{L^2}t}\ \sin\left(\frac{n\,\pi}{L}x\right)$$

Finally recall initial condition $u(x,0) = f(x)$. We simply force our solution to agree with this

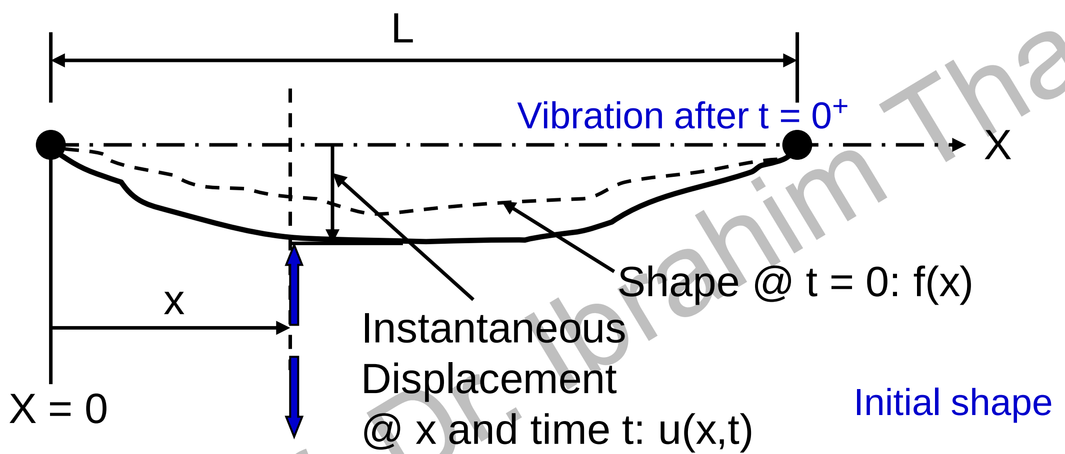
$$u(x;0) = \sum_{n=1}^{\infty} cn\sin\left(\frac{n\pi}{L}x\right)$$

which is called a Fourier sine series (FSS) with $\boldsymbol{c_n}\,'\boldsymbol{s}$ are given by the formula

$$\boldsymbol{c_n = \frac{2}{l}\int_0^l F(x)sin\frac{n\pi x}{l}dx}$$

Solution of partial differential equation for vibration of a string:

Consider a wave on a guitar string. In the simplest case $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$. The length of the string = L, and it is fixed at both ends at x = 0 and x = L, and as a result we know $0 \le x \le L$ and $t > 0$. BCs: $u(0,t) = u(L,t) = 0$, IC: $u(x,0) = f(x) and\ u_t(x,0) = 0)$. Solve the equation using separation of variable



The deflection of the string at x-distance from the end at x = 0 at time t into the vibration can be obtained by solving the following partial differential equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

in which $a = \sqrt{\frac{P}{m}}$ with P = tension in the string, and m = mass density of the string per unit length

Solution

$u(x,t) = X(x)\,T(t)$

$u_{xx} = X''(x)\,T(t)$ $\qquad$ $u_{tt} = X(x)\;T''(t)$

Now, substituting these expression into $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$. $\qquad$ $X(x)\;T''(t) = a^2\,X''(x)\,T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = k$$

$$\frac{X''(x)}{X(x)} = k \qquad X''(x) - k\,X(x) = 0$$

$$\frac{T''(t)}{a^2 T(t)} = k \qquad T''(t) - ka^2 T(t) = 0$$

Looking at the boundary conditions, we conclude if $k < 0 = -\lambda^2$ for case 3

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0$$

$$T''(t) + a^2\lambda^2 T(t) = 0$$

and thus we get the general solution for *X(x)* and *T(t)* of

$$X(x) = A\cos(\lambda x) + B\sin((\lambda x)$$

$$T(t) = C\cos(\lambda a t) + D\sin((\lambda a t)$$

Because we have knew that $u(x,t) = X(x)T(t)$

$X(x) = A\cos(\lambda x) + B\sin((\lambda x)$ $\qquad$ $T(t) = C\cos(\lambda at) + D\sin((\lambda at)$

$$thus\ u(x;\ t) = [A\cos\ (\lambda x) + B\sin(\lambda x)]\,[C\cos(\lambda at) + D\sin((\lambda at)]$$

where A, B, C, and D are arbitrary constants need to be determined from initial and boundary conditions given in above equation.

The *B.C* and *IC* are

$X(x) = A\cos(\lambda x) + B\sin((\lambda x)$ $\qquad$ $T(T) = C\cos(\lambda at) + D\sin((\lambda at)$

X(0) = 0 $\qquad$ X(L) = 0 $\qquad$ T(0) = f(x) $\qquad$ $\left.\dfrac{dT(t)}{dt}\right|_{t=0} = 0$

Determination of arbitrary constants:

•Let us start with the solution: $X(x) = A\cos(\lambda x) + B\sin((\lambda x)$

From B.C X(0) = 0: $\longrightarrow$ A cos (λ*0) + B sin (λ * 0) = 0, which means that A = 0

Now, from B.C): X(L) = 0: $\longrightarrow$ X(L) = 0 = B Sin(λL)

At this point, B = 0, or Sin (λ L) = 0 from the above relationship. A careful look at these choices will conclude that B ≠ 0, which leads to:

$Sin\ (\lambda L) = 0$ $\longrightarrow$ $\lambda L = n\pi$ $\quad$ *or* $\quad$ $\lambda_n = \dfrac{n\pi}{L}$

Now, if we substitute the solution of X(x) and $\lambda_n = \frac{n\pi}{L}$ in u (t, x),

$$u(x,t) = [A\cos(\lambda x) + B\sin(\lambda x)]\,[C\cos(\lambda \alpha t) + D\sin((\lambda \alpha t)]$$

we get:

$$u(x,t) = B\sin\left(\frac{n\pi}{L}x\right)\left[C\cos\left(\frac{n\pi}{L}at\right) + D\sin\left(\frac{n\pi}{L}at\right)\right] \qquad (n = 1, 2, 3,,\ldots\ldots)$$

By combining constants B, C and D in the above expression, we obtain *u(x,t)* to :

$$u(x,t) = \sin\left(\frac{n\,\pi}{L}x\right)\left[C_n\cos\left(\frac{n\pi}{L}at\right) + D_n\sin\left(\frac{n\,\pi}{L}at\right)\right] \qquad (n = 1, 2, 3,,\ldots\ldots)$$

We are now ready to use the two initial conditions to determine constants C_n and D_n in the above expression:

Let us first look at the condition : $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$

$$\left|\frac{\partial u(x,t)}{\partial\, t}\right|_{t=0} = 0 = \frac{n\,a\,\pi}{L}\sin\left(\frac{n\,\pi}{L}x\right)\left|\left[-C_n \sin\left(\frac{n\ \pi}{L}at\right) + D_n \cos\left(\frac{n\,\pi}{L}at\right)\right]\right|_{t=0}$$

But since $Sin\frac{n\pi}{L}x \neq 0$ $\longrightarrow$ $D_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\,\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$$

In order to determine constant coefficients C_n in previous equation :

The last remaining condition of ***u(x , o) = f(x)*** will be used for this purpose, in which *f(x)* is the initial shape of the string.

Thus, by letting $\boldsymbol{u(x,0) = f(x)}$, we will have:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)$$

which is called a Fourier sine series (FSS)

The coefficient C_n of the above Fourier series is:

$$\boldsymbol{C_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx}$$

The complete solution of the amplitude of vibrating string $u(x,t)$ becomes:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{L}\left(\int_0^L f(x)\,Sin\frac{n\pi x}{L}dx\right) Cos\frac{n\pi at}{L}\,Sin\frac{n\pi x}{L}$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Note that the equation has **no** dependence on time, just on the spatial variables x, y. This means that Laplace's Equation describes **steady state** situations such as:

- **steady state temperature distributions**
- **steady state stress distributions**
- **steady state flows, for example in a cylinder, around a corner,**

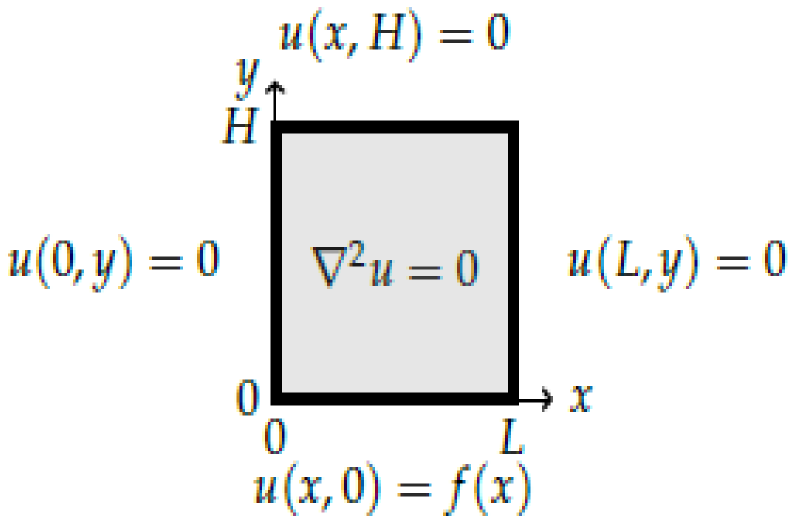
Example

Solve using variable separation, the temperature equilibrium distribution in rectangular plate.

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < L, \qquad 0 < y < H,$$

$$u(x,0) = f(x), \qquad u(x,H) = 0,$$

$$u(0,y) = u(L,y) = 0,$$



Solution

$$u(x,y) = X(x)\,Y(y)$$

$$X''(x)\,Y(y) + Y''(y)\,X(x) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2$$

$$X''(x) + \lambda^2 X(x) = 0 \qquad Y''(x) - \lambda^2 Y(y) = 0$$

Looking at the boundary conditions, we conclude if $k < 0 = -\lambda^2$ for case 3 and thus we get the general solution for u(x, y) = $X(x)$ and $Y(Y)$ of

$$Y''(x) - \lambda^2 Y(y) = 0$$

$$X(x) = A\cos(\lambda x) + B\sin((\lambda x)$$

$$Y(y) = Ce^{\lambda y} + De^{-\lambda y}$$

$$X''(x) + \lambda^2 X(x) = 0$$

u(x, y) = $[A\cos(\lambda x) + B\sin((\lambda x)]\,(Ce^{\lambda y} + De^{-\lambda y})$

where A, B, C, and D are arbitrary constants need to be determined from initial and boundary conditions given in above equation.

The *B.C* and *IC* are

$$X(x) = A\cos(\lambda x) + B\sin((\lambda x)$$

$$Y(y) = Ce^{\lambda y} + De^{-\lambda y}$$

$X(0) = 0$ $X(L) = 0$ $u(x,0) = f(x)$ $u(x,H) = 0$

- Let us start with the solution: $X(x) = A\cos(\lambda x) + B\sin((\lambda x)$

From B.C X(0) = 0: A cos (λ*0) + B sin (λ * 0) = 0, which means that A = 0

Now, from B.C): X(L) = 0: X(L) = 0 = B Sin(λL)

At this point, B = 0, or Sin (λ L) = 0 from the above relationship. A careful look at these choices will conclude that B $\neq$ 0, which leads to:

$Sin\ (\lambda L) = 0$ $\longrightarrow$ $\lambda L = n\pi$ *or* $\lambda_n = \frac{n\pi}{L}$

$X(x)n = B_n \sin\left(\frac{n\,\pi}{L}x\right)$ $\quad$ $X(x) = A\cos(\lambda x) + B\sin((\lambda x)$

$Y(y) = Ce^{\frac{n\,\pi}{L}y} + De^{-\frac{n\,\pi}{L}y}$

We are now ready to use the two boundary conditions to determine constants C and D in the above expression:

The boundary condition $\quad$ u(x, H) = 0

Therefore $\quad Ce^{\frac{n\,\pi}{L}H} + De^{-\frac{n\,\pi}{L}H} = 0$

Thus $\quad D = -C\,e^{2\frac{n\,\pi}{L}H}$

Therefore, Y(y) becomes

$$Y(y) = Ce^{\frac{n\,\pi}{L}y} - C\,e^{2\frac{n\,\pi}{L}H}e^{-\frac{n\,\pi}{L}y}$$

$$Y(y) = C\,e^{\frac{n\,\pi}{L}H}(e^{-\frac{n\,\pi}{L}H}\,e^{\frac{n\,\pi}{L}y} - e^{\frac{n\,\pi}{L}H}e^{-\frac{n\,\pi}{L}y})$$

$$Y(y) = C\,e^{\frac{n\,\pi}{L}H}\,(e^{-\frac{n\,\pi}{L}(H-y)} - e^{\frac{n\,\pi}{L}(H-y)}$$

$$Y(y) = -2\,C\,e^{\frac{n\,\pi}{L}H}\,(\sinh\frac{n\,\pi}{L}(H-y)$$

Therefore, the solution of $u(x,y)$ becomes

$$u(x,y) = B_n \sin\left(\frac{n\pi}{L}x\right)(-2)\,C_n\, e^{\frac{n\pi}{L}H}\ (\sinh \frac{n\pi}{L}(H - y)$$

or

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\ (\sinh \frac{n\pi}{L}(H - y)$$

In order to determine constant coefficients b_n in previous equation :

The last remaining condition of ***u(x , o) = f(x)*** will be used for this purpose

$$f(x) = \sum_{n=1}^{\infty} b_n\,(\sinh \frac{n\pi}{L}H)\ \ \sin\left(\frac{n\pi}{L}x\right)$$

$put\ F_n = \ b_n(\sinh \frac{n\pi}{L}H)$, we get

$$f(x) = \sum_{n=1}^{\infty} F_n\ \sin\left(\frac{n\pi}{L}x\right)$$ which is called a Fourier sine series (FSS)

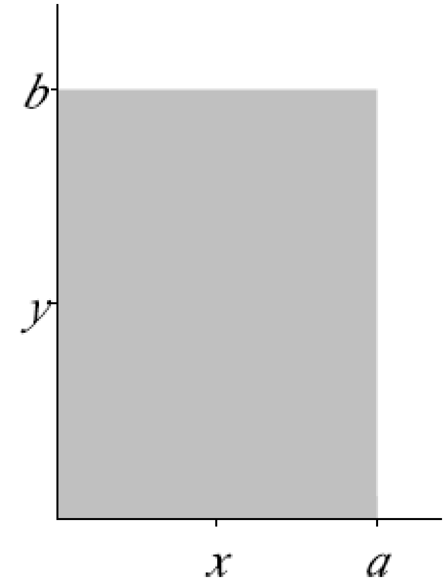
The coefficient $\boldsymbol{F_n}$ of the above Fourier series is: $\boldsymbol{F_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}\,dx}$

The two-dimensional wave equation

Consider a thin elastic membrane stretched tightly over a rectangular frame. Suppose the dimensions of the frame are $a \times b$ and that we keep the edges of the membrane fixed to the frame.

We let

$$u(x, y, t) = \text{deflection of membrane from equilibrium at position } (x, y) \text{ and time } t.$$



For a fixed t, the surface $z = u(x, y, t)$ gives the shape of the membrane at time t.

Under ideal assumptions (e.g. uniform membrane density, uniform tension, no resistance to motion, small deflection, etc.) one can show that u satisfies the **two dimensional wave equation**

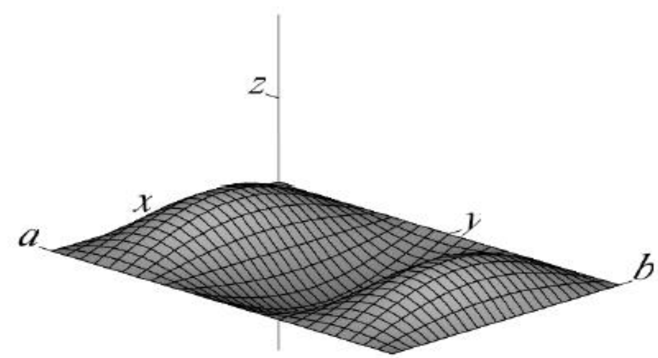
$$\boxed{u_{tt} = c^2 \nabla^2 u = c^2(u_{xx} + u_{yy})} \qquad (1)$$

for $0 < x < a$, $0 < y < b$.

Boundary conditions

$$u(0, y, t) = u(a, y, t) = 0, \qquad 0 \le y \le b,\ t \ge 0,$$
$$u(x, 0, t) = u(x, b, t) = 0, \qquad 0 \le x \le a,\ t \ge 0. \qquad (2)$$

$$u(x, y, 0) = f(x, y), \qquad (x, y) \in R,$$
$$u_t(x, y, 0) = g(x, y), \qquad (x, y) \in R, \qquad (3)$$



where $R = [0, a] \times [0, b]$.

Solve this problem using separation of variables.

Solution

$$u(x, y, t) = X(x)Y(y)T(t).$$

Firstly

Plugging this into the wave equation (1) we get

$$XYT'' = c^2\left(X''YT + XY''T\right).$$

If we divide both sides by c^2XYT this becomes

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Because the two sides are functions of different independent variables, they must be constant:

$$\frac{T''}{c^2T} = A = \frac{X''}{X} + \frac{Y''}{Y}.$$

The first equality becomes

$$T'' - c^2AT = 0.$$

The second can be rewritten as

$$\frac{X''}{X} = -\frac{Y''}{Y} + A.$$

Once again, the two sides involve unrelated variables, so both are constant:

$$\frac{X''}{X} = B = -\frac{Y''}{Y} + A.$$

If we let $C = A - B$ these equations can be rewritten as

$$X'' - BX = 0,$$
$$Y'' - CY = 0.$$

Secondly

The first boundary condition is

$$0 = u(0, y, t) = X(0)Y(y)T(t),\ 0 \leq y \leq b,\ t \geq 0.$$

Since we want nontrivial solutions only, we can cancel Y and T, yielding

$$X(0) = 0.$$

When we perform similar computations with the other three boundary conditions we also get

$$X(a) = 0,$$
$$Y(0) = Y(b) = 0.$$

There are no boundary conditions on T.

Fortunately, we have already solved the two boundary value problems for X and Y. The nontrivial solutions are

$$X_m(x) = \sin \mu_m x, \qquad \mu_m = \frac{m\pi}{a}, \qquad m = 1, 2, 3, \ldots$$
$$Y_n(y) = \sin \nu_n y, \qquad \nu_n = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

with separation constants $B = -\mu_m^2$ and $C = -\nu_n^2$.

Recall that T must satisfy

$$T'' - c^2 AT = 0$$

with $A = B + C = -\left(\mu_m^2 + \nu_n^2\right) < 0$. It follows that for any choice of m and n the general solution for T is

$$T_{mn}(t) = B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t,$$

where

$$\lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

These are the **characteristic frequencies** of the membrane.

Assembling our results, we find that for any *pair* $m, n \geq 1$ we have the **normal mode**

$$\boxed{\begin{aligned} u_{mn}(x, y, t) &= X_m(x) Y_n(y) T_{mn}(t) \\ &= \sin \mu_m x \sin \nu_n y \left(B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t\right) \end{aligned}}$$

where

$$\boxed{\mu_m = \frac{m\pi}{a}, \ \nu_n = \frac{n\pi}{b}, \ \lambda_{mn} = c\sqrt{\mu_m^2 + \nu_n^2}.}$$

Remarks:

- Note that the normal modes:
  - oscillate spatially with frequency μ_m in the x-direction,
  - oscillate spatially with frequency ν_n in the y-direction,
  - oscillate in time with frequency λ_{mn}.

- While μ_m and ν_n are simply multiples of π/a and π/b, respectively,

  λ_{mn} is not a multiple of any basic frequency.

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin \mu_m x \sin \nu_n y \left(B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t\right).}$$

Finally, we must determine the values of the coefficients B_{mn} and B^*_{mn} that are required so that our solution also satisfies the initial conditions (3). The first of these is

$$f(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

and the second is

$$g(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{mn} B^*_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y.$$

These are examples of **double Fourier series**.

Using the usual argument, it follows that *assuming* we can write

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y,$$

$$= \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx \tag{4}$$

Theorem

Suppose that $f(x,y)$ and $g(x,y)$ are C^2 functions on the rectangle $[0,a]\times[0,b]$. The solution to the wave equation (1) with boundary conditions (2) and initial conditions (3) is given by

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\mu_m x \sin\nu_n y\left(B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t\right)$$

where

$$\mu_m = \frac{m\pi}{a},\ \nu_n = \frac{n\pi}{b},\ \lambda_{mn} = c\sqrt{\mu_m^2+\nu_n^2},$$

and the coefficients B_{mn} and B_{mn}^* are given by

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx$$

and

$$B_{mn}^* = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dy\,dx.$$

Second-Order Linear Differential Equations

A **second-order linear differential equation** has the form

$$\boxed{1} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions. We saw in Section 7.1 that equations of this type arise in the study of the motion of a spring. In *Additional Topics: Applications of Second-Order Differential Equations* we will further pursue this application as well as the application to electric circuits.

In this section we study the case where $G(x) = 0$, for all x, in Equation 1. Such equations are called **homogeneous** linear equations. Thus, the form of a second-order linear homogeneous differential equation is

$$\boxed{2} \qquad P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

If $G(x) \neq 0$ for some x, Equation 1 is **nonhomogeneous** and is discussed in *Additional Topics: Nonhomogeneous Linear Equations*.

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

Proof Since y_1 and y_2 are solutions of Equation 2, we have

$$P(x)y_1'' + Q(x)y_1' + R(x)y_1 = 0$$

and
$$P(x)y_2'' + Q(x)y_2' + R(x)y_2 = 0$$

Therefore, using the basic rules for differentiation, we have

$$\begin{aligned}
P(x)y'' &+ Q(x)y' + R(x)y \\
&= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\
&= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\
&= c_1[P(x)y_1'' + Q(x)y_1' + R(x)y_1] + c_2[P(x)y_2'' + Q(x)y_2' + R(x)y_2] \\
&= c_1(0) + c_2(0) = 0
\end{aligned}$$

Thus, $y = c_1y_1 + c_2y_2$ is a solution of Equation 2.

The other fact we need is given by the following theorem, which is proved in more advanced courses. It says that the general solution is a linear combination of two **linearly independent** solutions y_1 and y_2. This means that neither y_1 nor y_2 is a constant multiple of the other. For instance, the functions $f(x) = x^2$ and $g(x) = 5x^2$ are linearly dependent, but $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent.

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Theorem 4 is very useful because it says that if we know *two* particular linearly independent solutions, then we know *every* solution.

In general, it is not easy to discover particular solutions to a second-order linear equation. But it is always possible to do so if the coefficient functions P, Q, and R are constant functions, that is, if the differential equation has the form

5
$$\boxed{ay'' + by' + cy = 0}$$

where a, b, and c are constants and $a \neq 0$.

It's not hard to think of some likely candidates for particular solutions of Equation 5 if we state the equation verbally. We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus, $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

6 $$\boxed{ar^2 + br + c = 0}$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

7 $$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

CASE I □ $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions of Equation 5. (Note that e^{r_2x} is not a constant multiple of e^{r_1x}.) Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

SOLUTION The auxiliary equation is

$$r^2 + r - 6 = (r - 2)(r + 3) = 0$$

whose roots are $r = 2, -3$. Therefore, by (8) the general solution of the given differential equation is

$$y = c_1e^{2x} + c_2e^{-3x}$$

We could verify that this is indeed a solution by differentiating and substituting into the differential equation.

EXAMPLE 2 Solve $3\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

SOLUTION To solve the auxiliary equation $3r^2 + r - 1 = 0$ we use the quadratic formula:

$$r = \frac{-1 \pm \sqrt{13}}{6}$$

Since the roots are real and distinct, the general solution is

$$y = c_1e^{(-1+\sqrt{13})x/6} + c_2e^{(-1-\sqrt{13})x/6}$$

CASE II □ $b^2 - 4ac = 0$

In this case $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let's denote by r the common value of r_1 and r_2. Then, from Equations 7, we have

$$\boxed{9} \qquad r = -\frac{b}{2a} \qquad \text{so} \quad 2ar + b = 0$$

We know that $y_1 = e^{rx}$ is one solution of Equation 5. We now verify that $y_2 = xe^{rx}$ is also a solution:

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + 0(xe^{rx}) = 0 \end{aligned}$$

The first term is 0 by Equations 9; the second term is 0 because r is a root of the auxiliary equation. Since $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions, Theorem 4 provides us with the general solution.

> $\boxed{10}$ If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is
>
> $$y = c_1e^{rx} + c_2xe^{rx}$$

EXAMPLE 3 Solve the equation $4y'' + 12y' + 9y = 0$.

SOLUTION The auxiliary equation $4r^2 + 12r + 9 = 0$ can be factored as

$$(2r + 3)^2 = 0$$

so the only root is $r = -\frac{3}{2}$. By (10) the general solution is

$$y = c_1 e^{-3x/2} + c_2 x e^{-3x/2}$$

CASE III $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 of the auxiliary equation are complex numbers. (See Appendix I for information about complex numbers.) We can write

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers. [In fact, $\alpha = -b/(2a)$, $\beta = \sqrt{4ac - b^2}/(2a)$.] Then, using Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

from Appendix I, we write the solution of the differential equation as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} \\ &= C_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + C_2 e^{\alpha x}(\cos\beta x - i\sin\beta x) \\ &= e^{\alpha x}[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x] \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \end{aligned}$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$. This gives all solutions (real or complex) of the differential equation. The solutions are real when the constants c_1 and c_2 are real. We summarize the discussion as follows.

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

EXAMPLE 4 Solve the equation $y'' - 6y' + 13y = 0$.

SOLUTION The auxiliary equation is $r^2 - 6r + 13 = 0$. By the quadratic formula, the roots are

$$r = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

By (11) the general solution of the differential equation is

$$y = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

EXAMPLE 5 Solve the initial-value problem

$$y'' + y' - 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$$

SOLUTION From Example 1 we know that the general solution of the differential equation is

$$y(x) = c_1e^{2x} + c_2e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2c_1e^{2x} - 3c_2e^{-3x}$$

To satisfy the initial conditions we require that

12 $$y(0) = c_1 + c_2 = 1$$

13 $$y'(0) = 2c_1 - 3c_2 = 0$$

From (13) we have $c_2 = \frac{2}{3}c_1$ and so (12) gives

$$c_1 + \tfrac{2}{3}c_1 = 1 \qquad c_1 = \tfrac{3}{5} \qquad c_2 = \tfrac{2}{5}$$

Thus, the required solution of the initial-value problem is

$$y = \tfrac{3}{5}e^{2x} + \tfrac{2}{5}e^{-3x}$$

EXAMPLE 6 Solve the initial-value problem

$$y'' + y = 0 \qquad y(0) = 2 \qquad y'(0) = 3$$

SOLUTION The auxiliary equation is $r^2 + 1 = 0$, or $r^2 = -1$, whose roots are $\pm i$. Thus $\alpha = 0$, $\beta = 1$, and since $e^{0x} = 1$, the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Since $$y'(x) = -c_1 \sin x + c_2 \cos x$$

the initial conditions become

$$y(0) = c_1 = 2 \qquad y'(0) = c_2 = 3$$

Therefore, the solution of the initial-value problem is

$$y(x) = 2\cos x + 3\sin x$$